A Probabilistic Weyl-law for Perturbed Berezin-Toeplitz Operators

Izak Oltman
University of California, Berkeley
Department of Mathematics

Preliminaries

These preliminaries can be found in Le Floch's textbook *A Brief Introduction to Berezin–Toeplitz Operators on Compact Kähler Manifolds* [Le Floch 2018].

1. A **Kähler manifold** is a complex manifold X with with three compatible structures: a complex structure, a symplectic form σ and a Riemannian metric.
2. We assume on X there is a **holomorphic line bundle** L with a positively curved Hermitian metric h. Locally, on each trivialization, there exists a smooth, real-valued function φ such that over the fiber $x \in X$.
$$h(u, v) = e^{-\varphi(x)} u\bar{v}. \tag{1}$$
3. The locally defined φ is called the **Kähler potential**, and is related to the globally defined sympletic form σ by $i\partial\bar{\partial}\varphi = \sigma$.
4. We let L^N be the N^{th} tensor power of the line bundle L which has Hermitian metric h^N.
5. We define an L^2 structure of smooth sections on L^N by defining
$$\langle u, v\rangle = \int_X u\bar{v}e^{-N\varphi}\left(\frac{\sigma^{\wedge d}}{d!}\right). \tag{2}$$
6. Let $L^2(X, L^N)$ be the space of smooth sections of L^N which have finite L^2 norm. Let $H^0(X, L^N)$ be the space of holomorphic sections in $L^2(X, L^N)$. This turns out to be a finite dimensional space.
7. Let Π_N be the orthogonal projection from $L^2(X, L^N)$ to $H^0(X, L^N)$, this is called the **Bergman kernel**.
8. Given a smooth function f on X, the **Toeplitz operator** $T_N f$ acts on holomorphic sections $u \in H^0(X, L^N)$ by
$$T_N f(u) = \Pi_N(fu). \tag{3}$$
9. For each $N \in \mathbb{N}$, $T_N f$ is a finite rank operator mapping $H^0(X, L^N)$ to itself.

Theorem 1: Main Result

Given:

- *A compact, connected, d–dimensional Kähler manifold (X, σ) with a holomorphic line bundle L with positively curved Hermitian metric.*
- *A $f \in C^\infty(M, \mathbb{C})$.*
- *$\mathcal{G}$, a finite rank operator mapping $H^0(X, L^N)$ to itself whose matrix representation with respect to a fixed basis is a random matrix whose entries are i.i.d. complex Gaussian random variables with mean 0 and variance 1.*

Then for any open $\Omega \subset \mathbb{C}$:
$$\left(\frac{2\pi}{N}\right)^d \#\{\mathrm{Spec}(T_N f + N^{-d}\mathcal{G}) \cap \Omega\} \xrightarrow{N\to\infty} f^*\left(\frac{\sigma^\wedge d}{d!}\right)(\Omega) \tag{4}$$
almost surely.

Vogel's Theorem for Quantizations of the Torus

This result is a generalization of a theorem proven by Martin Vogel [Vogel 2020]. Vogel studied a different, but related, quantization. In this case functions on the torus of real dimension $2d$ are associated to $N^d \times N^d$ matrices.

As an example, under this quantization, the function $f(x, \xi) = \cos(x) + i\cos(\xi)$ is associated to the family of matrices
$$f_N = \begin{pmatrix} \cos(2\pi/N) & i/2 & 0 & 0 & \cdots & i/2 \\ i/2 & \cos(4\pi/N) & i/2 & 0 & \cdots & 0 \\ 0 & i/2 & \cos(6\pi/N) & i/2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & i/2 & \cos(2(N-1)\pi/N) & i/2 \\ i/2 & 0 & \cdots & 0 & i/2 & \cos(2\pi) \end{pmatrix}. \tag{5}$$

Below is a plot of the spectrum of f_N with and without a random perturbation.

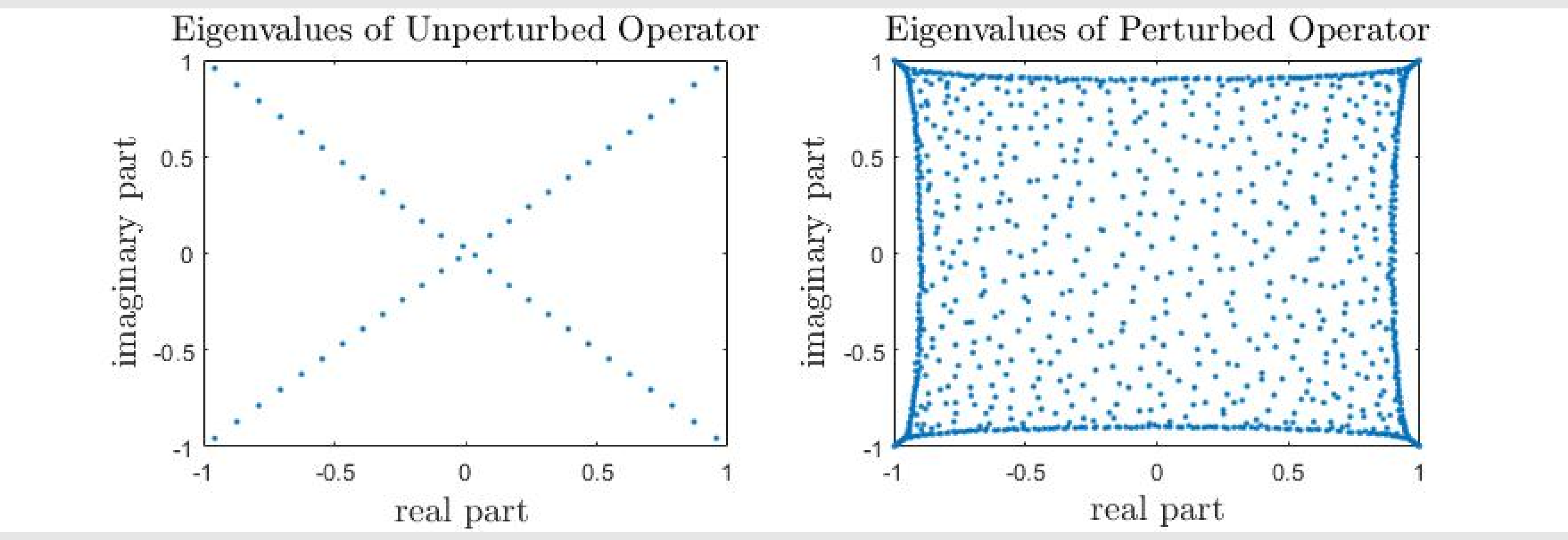

Figure 1. Left is f_N with no perturbation (this is known as the Scottish flag operator). Right is f_N with a small random perturbation.

A consequence [Vogel 2020] is that the in any region $\Omega \subset \mathbb{C}$ the number of eigenvalues in Ω of the perturbed matrix is
$$\approx m(\{(x, \xi) \in \mathbb{T}^{2d} : \cos(x) + i\cos(\xi) \in \Omega\}) \tag{6}$$
as $N \to \infty$ almost surely. See [Vogel 2020] for more precise results).

Numerical Verification of Vogel's Theorem

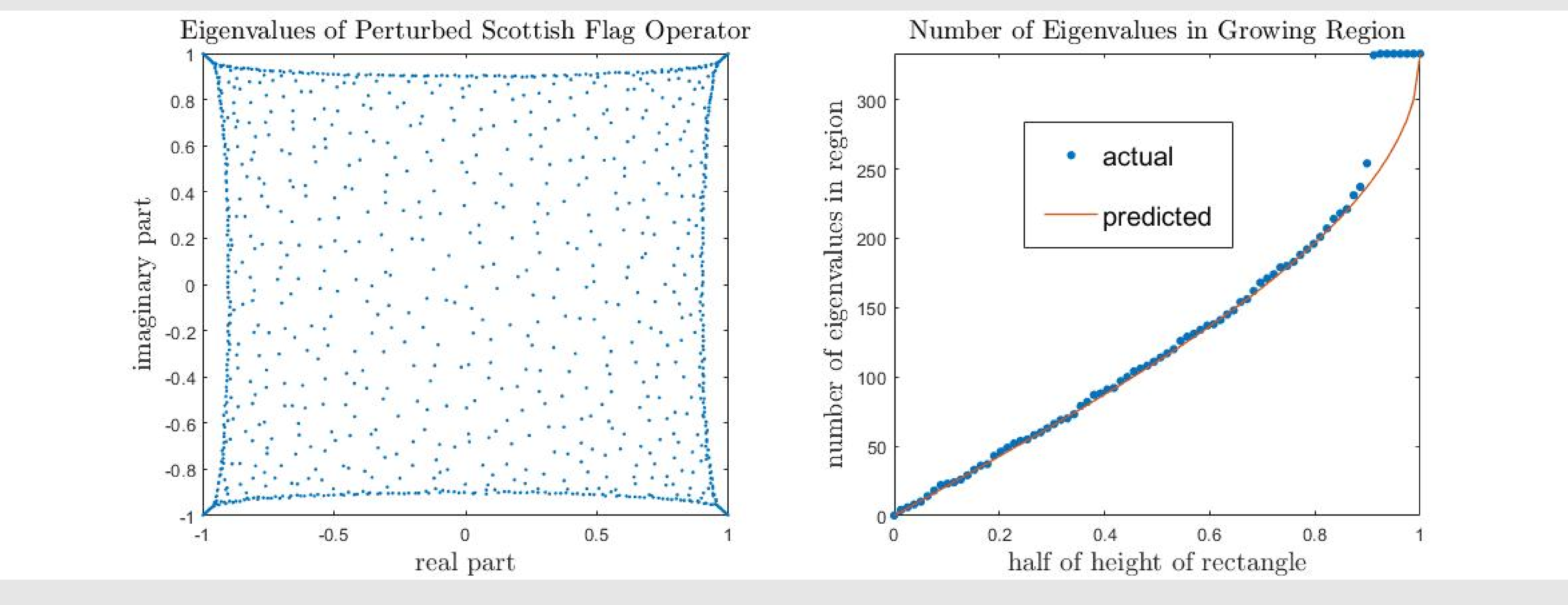

Figure 2. Left: the eigenvalues of the f_N with a small random perturbation added. Right: the number of eigenvalues in a rectangle centered at the origin of fixed width and increasing height as predicted by Vogel's theorem and computed numerically.

Example: $\mathbb{CP}^1$

Consider $\mathbb{CP}^1$ (complex projective space of complex dimension 1). This is a Kähler manifold. If we take the trivial line bundle, and take the dual, then we get a positively curved Hermitian metric, for which we can take tensor powers. We can consider a single chart, in which case we can write:

1. $\sigma(x) = (1 + |x|^2)^{-2}dx \wedge d\bar{x}$ (symplectic form),
2. $\varphi(x) = \log(1 + |x|^2)$ (Kähler potential),
3. smooth sections can be identified with smooth functions on $\mathbb{C}$ on which we have the L^2 inner product
$$\langle f, g\rangle = \int_{\mathbb{C}} \frac{f(z)\overline{g(z)}}{(1 + |z|^2)^{N+2}} dm(z). \tag{7}$$
4. Then $H^0(X, L^N)$ has an orthonormal basis $c_k z^k$ for $k \le N$ with
$$c_k = \sqrt{\binom{N}{k}\frac{N+1}{\pi}} \tag{8}$$
5. The Bergman kernel is
$$\Pi_N(z, w) = \sum_{m=0}^{N} e_m(z)\overline{e_m(w)} = \cdots = \frac{N+1}{\pi}(1 + z\bar{w})^N. \tag{9}$$
6. If f is a smooth function on $\mathbb{CP}^1$, then in coordinates
$$T_N f(u)(z) = \int_{\mathbb{C}} \Pi_N(z, w) f(w) u(w) \frac{dm(w)}{(1 + |w|^2)^{N+2}} \tag{10}$$

Numerics for $\mathbb{CP}^1$

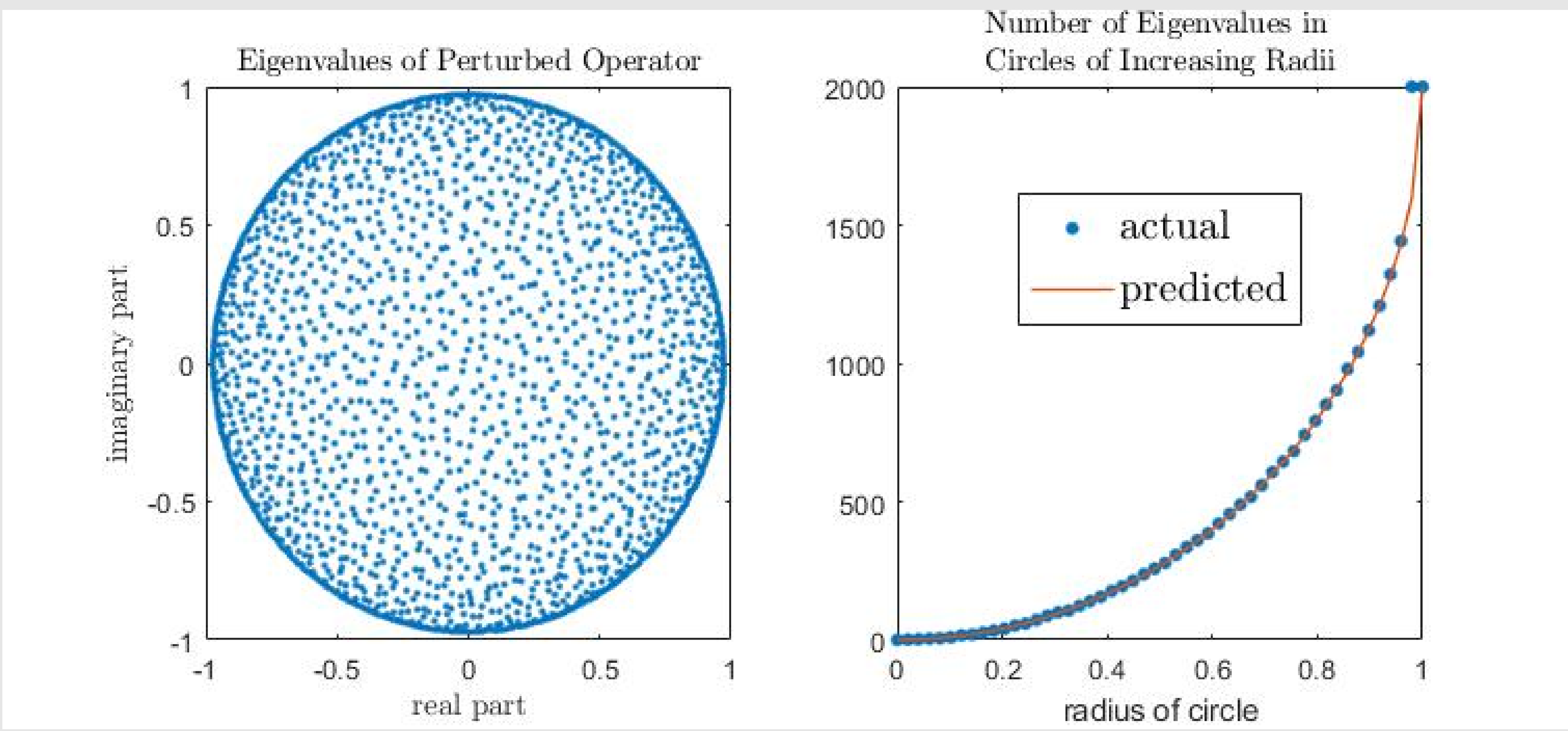

Figure 3. Identifying $\mathbb{CP}^1$ with the 2-sphere in $\mathbb{R}^3$ with coordinates x_1, x_2, x_3, we consider the quantization of $f = x_1 + ix_2$. On the left is the spectrum of $T_N f$ for $N = 2000$ with a random perturbation. On the right is the predicted number of eigenvalues within circles centered at the origin of increasing radii plotted against the predicted number from my result.

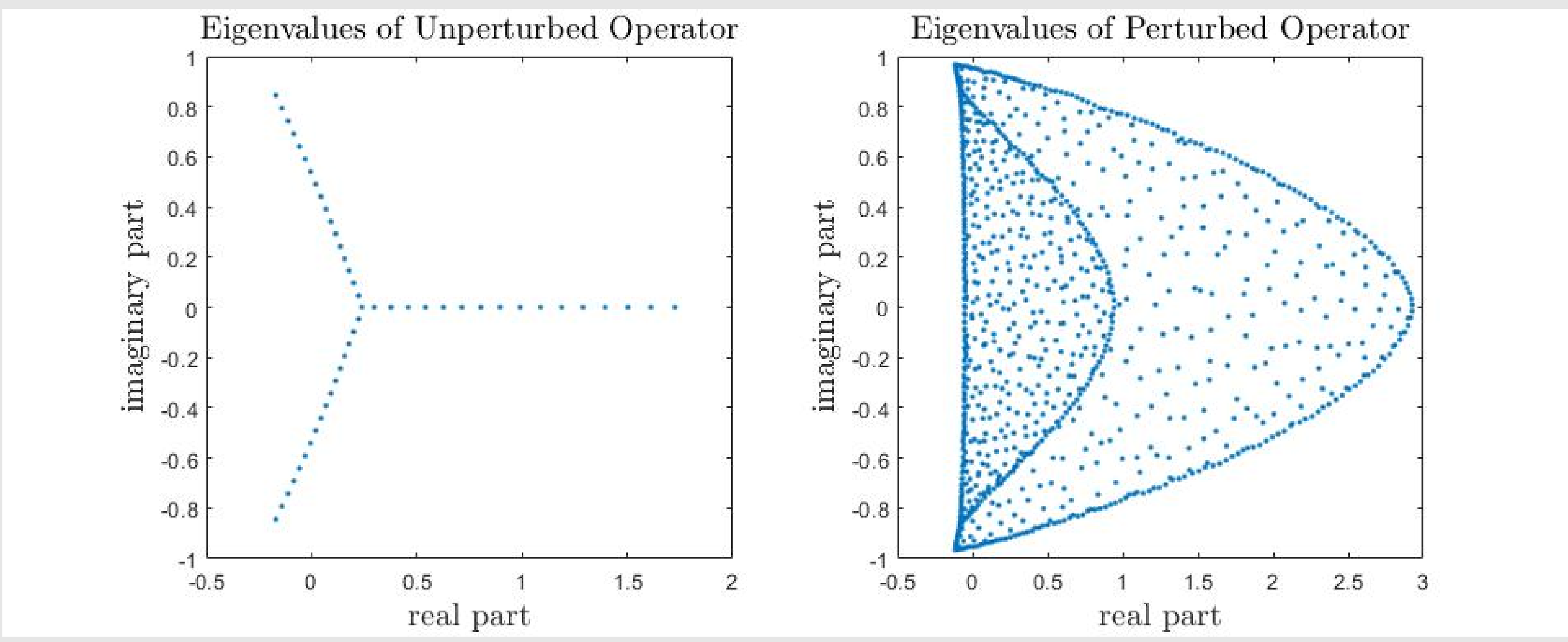

Figure 4. With the same identification as above, we consider the quantization of $f = x_1 + 2x_1^2 + x_2$. On the left is the spectrum of $T_N f$ with $N = 50$, and on the right is the spectrum for $N = 1000$ with random perturbation. This is an analogue of the Scottish flag operator.

Sketch Of Proof

1. By Borel-Cantelli and logarithmic potentials, it is enough to bound
$$\mathbb{P}(|N^{-d}\log|\det(T_N f + \delta\mathcal{G})| - \fint \log|z - f(x)|d\mu_d(x)| > N^{-\gamma}) \tag{11}$$
for some $\gamma > 0$
2. Set up a Grushin problem to control this.
3. Main obstruction is to estimate $\psi(N^\delta T_N f)$ as a Toeplitz operator for $\delta \in (0, 1)$.
4. Require developing an **exotic calculus** for Toeplitz operators.

Exotic Calculus

Consider the Kähler manifold $\mathbb{C}$. In this case define
$$S_\delta(m) = \{f \in C^\infty(\mathbb{C}) : |\partial^\alpha f| \le C_\alpha N^{\delta|\alpha|} m\} \tag{12}$$
for $\delta \in [0, 1/2)$ and m an order function. Then we need to show that if $f \in S_\delta(m_1)$ and $g \in S_\delta(m_2)$ then there exists $h \in S_\delta(m_3)$ such that
$$T_N f \circ T_N g = T_N h + \mathcal{O}(N^{-\infty}). \tag{13}$$
The main idea is to obtain an asymptotic expansion of the Schwartz kernel of the Toeplitz operators with an appropriate remainder.

This involves applying the method of complex stationary phase (developed by Melin and Sjöstrand [Melin and Sjöstrand 1975]). This involves almost analytically extending functions in $S_\delta(m)$ which obey weaker $\bar{\partial}$ estimates. This has to be be carefully handled to obtain an appropriate asymptotic expansion.

References

- Vogel, M. (2020). "Almost Sure Weyl Law for Quantized Tori". In: *Communications in Mathematical Physics*.
- Le Floch, Y. (2018). *A Brief Introduction to Berezin–Toeplitz Operators on Compact Kähler Manifolds*. Springer International Publishing.
- Melin, A. and J. Sjöstrand (1975). "Fourier integral operators with complex-valued phase functions". In: *Fourier Integral Operators and Partial Differential Equations*. Springer Berlin Heidelberg.